

GLOBAL SOLUTIONS FOR A HYPERBOLIC-PARABOLIC SYSTEM OF CHEMOTAXIS

RAFAEL GRANERO-BELINCHÓN

ABSTRACT. We study a hyperbolic-parabolic model of chemotaxis in dimensions one and two. In particular, we prove the global existence of classical solutions in certain dissipation regimes.

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1. INTRODUCTION

In this note we study the following system of partial differential equations

$$\begin{aligned} (1) \quad & \partial_t u = -\Lambda^\alpha u + \nabla \cdot (uq), \text{ for } x \in \mathbb{T}^d, t \geq 0, \\ (2) \quad & \partial_t q = \nabla f(u), \text{ for } x \in \mathbb{T}^d, t \geq 0, \end{aligned}$$

where u is a non-negative scalar function, q is a vector in \mathbb{R}^d , \mathbb{T}^d denotes the domain $[-\pi, \pi]^d$ with periodic boundary conditions, $d = 1, 2$ is the dimension, $f(u) = u^2/2$, $0 < \alpha \leq 2$ and $(-\Delta)^{\alpha/2} = \Lambda^\alpha$ is the fractional Laplacian.

This system was proposed by Othmers & Stevens [21] based on biological considerations as a model of tumor angiogenesis. In particular, in the previous system, u is the density of vascular endothelial cells and $q = \nabla \log(v)$ where v is the concentration of the signal protein known as vascular endothelial growth factor (VEGF) (see Bellomo, Li, & Maini [1] for more details on tumor modelling). Similar hyperbolic-dissipative systems arise also in the study of compressible viscous fluids or magnetohydrodynamics (see S. Kawashima [8] and the references therein).

Equation (1) appears as a singular limit of the following Keller-Segel model of aggregation of the slime mold *Dictyostelium discoideum* [9] (see also Patlak [20])

$$(3) \quad \begin{cases} \partial_t u = -\Lambda^\alpha u - \chi \nabla \cdot (u \nabla G(v)), \\ \partial_t v = \nu \Delta v + (f(u) + \lambda)v, \end{cases}$$

when $G(v) = \log(v)$ and the diffusion of the chemical is negligible, *i.e.* $\nu \rightarrow 0$.

Similar equations arising in different context are the Majda-Biello model of Rossby waves [18] or the magnetohydrodynamic-Burgers system proposed by Fleischer & Diamond [3].

Most of the results for (1) corresponds to the case where $d = 1$. Then, when the diffusion is local *i.e.* $\alpha = 2$, (1) has been studied by many different research groups. In particular, Fan & Zhao [2], Li & Zhao [13], Mei, Peng & Wang [19], Li, Pan & Zhao [12], Jun, Jixiong, Huijiang & Changjiang [7] Li & Wang [16] and Zhang & Zhu [25] studied the system (1) when $\alpha = 2$ and $f(u) = u$ under different boundary conditions (see also the works by Jin, Li & Wang [6], Li, Li & Wang [14], Wang & Hillen [22] and Wang, Xiang & Yu [23]). The case with general $f(u)$ was studied by Zhang, Tan & Sun [26] and Li & Wang [17].

Equation (1) in several dimensions has been studied by Li, Li & Zhao [11], Hau [5] and Li, Pan & Zhao [15]. There, among other results, the global existence for small initial data in H^s , $s > 2$ is proved.

To the best of our knowledge, the only result when the diffusion is non-local, *i.e.* $0 < \alpha < 2$, is [4]. In that paper we obtained appropriate lower bounds for the fractional Fisher information and, among other results, we proved the global existence of weak solution for $f(u) = u^r/r$ and $1 \leq r \leq 2$.

In this note, we address the existence of classical solutions in the case $0 < \alpha \leq 2$. This is a challenging issue due to the hyperbolic character of the equation for q . In particular, u verifies a transport equation where the velocity q is one derivative more singular than u (so $\nabla \cdot (uq)$ is two derivatives less regular than u).

2. STATEMENT OF THE RESULTS

For the sake of clarity, let us state some notation: we define the mean as

$$\langle g \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} g(x) dx.$$

Also, from this point onwards, we write H^s for the L^2 -based Sobolev space of order s endowed with the norm

$$\|u\|_{H^s}^2 = \|u\|_{L^2}^2 + \|u\|_{\dot{H}^s}^2, \quad \|u\|_{\dot{H}^s} = \|\Lambda^s u\|_{L^2}.$$

For $\beta \geq 0$, we consider the following energies E_β and dissipations D_β ,

$$(4) \quad E_\beta(t) = \|u\|_{\dot{H}^\beta}^2 + \|q\|_{\dot{H}^\beta}^2, \quad D_\beta(t) = \|u\|_{\dot{H}^{\beta+\alpha/2}}^2.$$

Recall that the lower order norms verify the following energy balance [4]

$$(5) \quad \frac{1}{2} (\|u(t)\|_{L^2}^2 + \|q(t)\|_{L^2}^2) + \int_0^t \|u(s)\|_{\dot{H}^{\alpha/2}}^2 ds = \frac{1}{2} (\|u_0\|_{L^2}^2 + \|q_0\|_{L^2}^2).$$

2.1. On the scaling invariance. Notice that the equations (1)-(2) verify the following scaling symmetry: for every $\lambda > 0$

$$u_\lambda(x, t) = \lambda^{\alpha-1} u(\lambda x, \lambda^\alpha t), \quad q_\lambda(x, t) = \lambda^{\alpha-1} q(\lambda x, \lambda^\alpha t).$$

This scaling serves as a *zoom in* towards the small scales. We also know that

$$\|u(t)\|_{L^2}^2 + \|q(t)\|_{L^2}^2$$

is the strongest (known) quantity verifying a global-in-time bound. Then, in the one dimensional case, the L^2 norms of u and q are invariant under the scaling of the equations when $\alpha = 1.5$. That makes $\alpha = 1.5$ the critical exponent for the global estimates known. Equivalently, if we define the rescaled (according to the scaling of the strongest conserved quantity $\|u(t)\|_{L^2}^2 + \|q(t)\|_{L^2}^2$) functions

$$u_\gamma(x, t) = \gamma^{0.5} u(\gamma x, \gamma^\alpha t), \quad q_\gamma(x, t) = \gamma^{0.5} q(\gamma x, \gamma^\alpha t).$$

we have that u_γ and q_γ solve

$$\begin{aligned} \partial_t u_\gamma &= -\Lambda^\alpha u_\gamma + \gamma^{\alpha-1.5} \partial_x(u_\gamma q_\gamma), \\ \partial_t q_\gamma &= \gamma^{\alpha-1.5} u_\gamma \partial_x u_\gamma. \end{aligned}$$

Larger values of α form the subcritical regime where the diffusion dominates the drift in small scales. Smaller values of α form the supercritical regime where the drift might be dominant at small scales.

Similarly, the two dimensional case has critical exponent $\alpha = 2$.

Remark 1. Notice that the equations (1)-(2) where $f(u) = u$ have a different scaling symmetry but the same critical exponent $\alpha = 1.5$. In this case, the scaling symmetry is given by

$$u_\lambda(x, t) = \lambda^{2\alpha-2} u(\lambda x, \lambda^\alpha t), \quad q_\lambda(x, t) = \lambda^{\alpha-1} q(\lambda x, \lambda^\alpha t),$$

while the conserved quantity is $\|u(t)\|_{L^1} + \|q(t)\|_{L^2}^2/2$. Thus, if we define the rescaled (according to the scaling of the conserved quantity) functions

$$u_\gamma(x, t) = \gamma u(\gamma x, \gamma^\alpha t), \quad q_\gamma(x, t) = \gamma^{0.5} q(\gamma x, \gamma^\alpha t).$$

we have that u_γ and q_γ solve

$$\begin{aligned} \partial_t u_\gamma &= -\Lambda^\alpha u_\gamma + \gamma^{\alpha-1.5} \partial_x(u_\gamma q_\gamma), \\ \partial_t q_\gamma &= \gamma^{\alpha-1.5} \partial_x u_\gamma. \end{aligned}$$

A global existence result when α is the range $1.5 \leq \alpha < 2$ for the problem where $f(u) = u$ is left for future research.

2.2. Results in the one-dimensional case $d = 1$. One of our main results is

Theorem 1. Fix T an arbitrary parameter and let $(u_0, q_0) \in H^2(\mathbb{T}) \times H^2(\mathbb{T})$ be the initial data such that $0 \leq u_0$ and $\langle q_0 \rangle = 0$. Assume that $\alpha \geq 1.5$. Then there exist a unique global solution $(u(t), q(t))$ to problem (1) verifying

$$u \in L^\infty(0, T; H^2(\mathbb{T})) \cap L^2(0, T; H^{2+\alpha/2}(\mathbb{T})), q \in L^\infty(0, T; H^2(\mathbb{T})).$$

Furthermore, the solution is uniformly bounded in

$$(u, q) \in C([0, \infty), H^1(\mathbb{T})) \times C([0, \infty), H^1(\mathbb{T})).$$

In the case where the strength of the diffusion, α , is even weaker, we have the following global existence result for small data:

Theorem 2. *Fix T an arbitrary parameter and let $(u_0, q_0) \in H^2(\mathbb{T}) \times H^2(\mathbb{T})$ be the initial data such that $0 \leq u_0$ and $\langle q_0 \rangle = 0$. There exists \mathcal{C}_α such that if $1.5 > \alpha > 1$ and*

$$\|u_0\|_{\dot{H}^{\alpha/2}}^2 + \|q_0\|_{\dot{H}^{\alpha/2}}^2 \leq \mathcal{C}_\alpha$$

then there exist a unique global solution $(u(t), q(t))$ to problem (1) verifying

$$u \in L^\infty(0, T; H^2(\mathbb{T})) \cap L^2(0, T; H^{2+\alpha/2}(\mathbb{T})), q \in L^\infty(0, T; H^2(\mathbb{T})).$$

Furthermore, the solution verifies

$$\|u(t)\|_{\dot{H}^{\alpha/2}}^2 + \|q(t)\|_{\dot{H}^{\alpha/2}}^2 \leq \|u_0\|_{\dot{H}^{\alpha/2}}^2 + \|q_0\|_{\dot{H}^{\alpha/2}}^2.$$

Corollary 1. *Fix T an arbitrary parameter and let $(u_0, q_0) \in H^2(\mathbb{T}) \times H^2(\mathbb{T})$ be the initial data such that $0 \leq u_0$ and $\langle q_0 \rangle = 0$. Assume that $1 \geq \alpha \geq 0.5$ and*

$$\|u_0\|_{H^1}^2 + \|q_0\|_{H^1}^2 < \frac{4}{9C_S^2}$$

where C_S is defined in (7). Then there exist a unique global solution $(u(t), q(t))$ to problem (1) verifying

$$u \in L^\infty(0, T; H^2(\mathbb{T})) \cap L^2(0, T; H^{2+\alpha/2}(\mathbb{T})), q \in L^\infty(0, T; H^2(\mathbb{T})).$$

Furthermore, the solution verifies

$$\|u(t)\|_{H^1}^2 + \|q(t)\|_{H^1}^2 \leq \|u_0\|_{H^1}^2 + \|q_0\|_{H^1}^2.$$

2.3. Results in the two-dimensional case $d = 2$. In two dimensions the global existence read

Theorem 3. *Fix T an arbitrary parameter and let $(u_0, q_0) \in H^2(\mathbb{T}^2) \times H^2(\mathbb{T}^2)$ be the initial data such that $0 \leq u_0$, $\langle q_0 \rangle = 0$ and $\text{curl} q_0 = 0$. Assume that $\alpha = 2$. Then there exist a unique global solution $(u(t), q(t))$ to problem (1) verifying*

$$u \in L^\infty(0, T; H^2(\mathbb{T}^2)) \cap L^2(0, T; H^3(\mathbb{T}^2)), q \in L^\infty(0, T; H^2(\mathbb{T}^2)).$$

Furthermore, the solution is uniformly bounded in

$$(u, q) \in C([0, \infty), H^1(\mathbb{T}^2)) \times C([0, \infty), H^1(\mathbb{T}^2)).$$

Corollary 2. *Fix T an arbitrary parameter and let $(u_0, q_0) \in H^2(\mathbb{T}^2) \times H^2(\mathbb{T}^2)$ be the initial data such that $0 \leq u_0$, $\langle q_0 \rangle = 0$ and $\text{curl} q_0 = 0$. Assume that $2 > \alpha \geq 1$ and*

$$\|u_0\|_{\dot{H}^1}^2 + \|q_0\|_{\dot{H}^1}^2 < \mathcal{C}$$

where \mathcal{C} is a universal constant. Then there exist a unique global solution $(u(t), q(t))$ to problem (1) verifying

$$u \in L^\infty(0, T; H^2(\mathbb{T}^2)) \cap L^2(0, T; H^{2+\alpha/2}(\mathbb{T}^2)), q \in L^\infty(0, T; H^2(\mathbb{T}^2)).$$

Furthermore, the solution verifies

$$\|u(t)\|_{H^1}^2 + \|\nabla \cdot q(t)\|_{L^2}^2 \leq \|u_0\|_{H^1}^2 + \|\nabla \cdot q_0\|_{L^2}^2.$$

Remark 2. In the case where the domain is the one-dimensional torus, \mathbb{T} , local existence of solution for (1)-(2) was proved in [4] for a more general class of kinetic function $f(u)$. The local existence of solution for (1)-(2) the domain is the two-dimensional torus \mathbb{T}^d with $d = 2$ follows from the local existence result in [4] with minor modifications. Consequently, we will focus on obtaining global-in-time a priori estimates.

2.4. Discussion. Due to the hyperbolic character of the equation for q , prior available global existence results of classical solution for equation (1) impose several assumptions. Namely,

- either $d = 1$ and $\alpha = 2$ [26, 17],
- or $d = 2, 3$, $\alpha = 2$ and the initial data verifies some smallness condition on *strong* Sobolev spaces H^s , $s \geq 2$ [24, 27].

Our results removed some of the previous conditions. On the one hand, we prove global existence for arbitrary data in the cases $d = 1$ and $\alpha \geq 1.5$ and $d = 2$ and $\alpha = 2$. On the other hand, in the cases where we have to impose size restrictions on the initial data, the Sobolev spaces are bigger than H^2 (thus, the norm is weaker). Finally, let us emphasize that our results can be adapted to the case where the spatial domain is \mathbb{R}^d .

A question that remains open is the trend to equilibrium. From (5) is clear that the solution $(u(t), q(t))$ tends to the homogeneous state, namely $(\langle u_0 \rangle, 0)$. However, the rate of this convergence is not clear.

3. PROOF OF THEOREM 1

Step 1; H^1 estimate: Testing the first equation in (1) against $\Lambda^2 u$, integrating by parts and using the equation for q , we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u\|_{\dot{H}^1}^2 + \|\partial_x u\|_{\dot{H}^{\alpha/2}}^2 &= - \int_{\mathbb{T}} \partial_x(uq) \partial_x^2 u dx \\ &= \frac{1}{2} \int_{\mathbb{T}} \partial_x q (\partial_x u)^2 dx - \int_{\mathbb{T}} \partial_x q u \partial_x^2 u dx \\ &= \frac{1}{2} \int_{\mathbb{T}} \partial_x q (\partial_x u)^2 dx - \int_{\mathbb{T}} \partial_x q (\partial_t \partial_x q - (\partial_x u)^2) dx, \end{aligned}$$

so

$$\frac{1}{2} \frac{d}{dt} (\|u\|_{\dot{H}^1}^2 + \|q\|_{\dot{H}^1}^2) + \|u\|_{\dot{H}^{1+\alpha/2}}^2 = \frac{3}{2} \int_{\mathbb{T}} \partial_x q (\partial_x u)^2 dx.$$

Denoting

$$I = \frac{3}{2} \int_{\mathbb{T}} \partial_x q (\partial_x u)^2 dx,$$

and using Sobolev embedding and interpolation, we have that

$$(6) \quad I \leq \frac{3}{2} \|q\|_{\dot{H}^1} \|\partial_x u\|_{L^4}^2 \leq \frac{3}{2} C_S \|q\|_{\dot{H}^1} \|\partial_x u\|_{\dot{H}^{0.25}}^2,$$

where C_S is the constant appearing in the embedding

$$(7) \quad \|g\|_{L^4} \leq C_S \|g\|_{\dot{H}^{0.25}}.$$

Using the interpolation

$$H^{1+\alpha/2} \subset H^{1.25} \subset H^{\alpha/2},$$

and Poincaré inequality (if $\alpha > 1.5$) we conclude

$$I \leq c\|q\|_{\dot{H}^1}\|\Lambda^{\alpha/2}u\|_{L^2}\|u\|_{\dot{H}^{1+\alpha/2}},$$

Using (4), we have that

$$\frac{d}{dt}E_1 + D_1 \leq c\|u\|_{\dot{H}^{\alpha/2}}^2 E_1.$$

Using Gronwall's inequality and the estimate (5), we have that

$$\sup_{0 \leq t < \infty} E_1(t) \leq C(\|u_0\|_{H^1}, \|q_0\|_{H^1}),$$

$$\int_0^T D_1(s)ds \leq C(\|u_0\|_{H^1}, \|q_0\|_{H^1}, T), \forall 0 < T < \infty.$$

Step 2; H^2 estimate: Now we prove that the solutions satisfying the previous bounds for E_1 and D_1 also satisfy the corresponding estimate in H^2 . We test the equation for u against $\Lambda^4 u$. We have that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u\|_{\dot{H}^2}^2 + \|u\|_{\dot{H}^{2+\alpha/2}}^2 &= - \int_{\mathbb{T}} \partial_x^2(uq) \partial_x^3 u dx \\ &= \int_{\mathbb{T}} \partial_x q \frac{5(\partial_x^2 u)^2}{2} dx - \int_{\mathbb{T}} \partial_x^2 q (\partial_t \partial_x^2 q - 5\partial_x u \partial_x^2 u) dx, \end{aligned}$$

so

$$\frac{1}{2} \frac{d}{dt} (\|u\|_{\dot{H}^2}^2 + \|q\|_{\dot{H}^2}^2) + \|u\|_{\dot{H}^{2+\alpha/2}}^2 = \frac{5}{2} \int_{\mathbb{T}} \partial_x q (\partial_x^2 u)^2 dx + 5 \int_{\mathbb{T}} \partial_x^2 q \partial_x^2 u \partial_x u dx.$$

We define

$$J_1 = \frac{5}{2} \int_{\mathbb{T}} \partial_x q (\partial_x^2 u)^2 dx, \quad J_2 = 5 \int_{\mathbb{T}} \partial_x^2 q \partial_x^2 u \partial_x u dx.$$

Then, we have that

$$J_1 \leq c\|\partial_x q\|_{L^\infty} \|\partial_x^2 u\|_{L^2}^2 \leq c\|\partial_x^2 q\|_{L^2}^{0.5} \|u\|_{\dot{H}^{1+\alpha/2}}^\alpha \|u\|_{\dot{H}^{2+\alpha/2}}^{2-\alpha},$$

so, using Young's inequality,

$$J_1 \leq c\|\partial_x^2 q\|_{L^2}^{\frac{1}{\alpha}} \|u\|_{\dot{H}^{1+\alpha/2}}^2 + \frac{1}{4} \|u\|_{\dot{H}^{2+\alpha/2}}^2.$$

Similarly, using Poincaré inequality and $\alpha \geq 0.5$,

$$J_2 \leq c\|\partial_x u\|_{L^4} \|\partial_x^2 u\|_{L^4} \|\partial_x^2 q\|_{L^2} \leq c\|u\|_{\dot{H}^{1+\alpha/2}} \|u\|_{\dot{H}^{2+\alpha/2}} \|\partial_x^2 q\|_{L^2},$$

and

$$J_2 \leq c\|u\|_{\dot{H}^{1+\alpha/2}}^2 \|\partial_x^2 q\|_{L^2}^2 + \frac{1}{4} \|u\|_{\dot{H}^{2+\alpha/2}}^2.$$

Finally,

$$\frac{d}{dt} E_2(t) + D_2(t) \leq c\|u\|_{\dot{H}^{1+\alpha/2}}^2 (E_2(t) + 1)$$

and we conclude using Gronwall's inequality.

4. PROOF OF THEOREM 2

Step 1; $H^{\alpha/2}$ estimate: Testing the first equation in (1) against $\Lambda^\alpha u$, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u\|_{\dot{H}^{\alpha/2}}^2 + \|u\|_{\dot{H}^\alpha}^2 &= \int_{\mathbb{T}} \partial_x(uq) \Lambda^\alpha u dx \\ &= - \int_{\mathbb{T}} \Lambda^\alpha(uq) \partial_x u dx \\ &= - \int_{\mathbb{T}} (\Lambda^\alpha(uq) - u \Lambda^\alpha q) \partial_x u dx - \int_{\mathbb{T}} \Lambda^\alpha q u \partial_x u dx, \end{aligned}$$

so

$$\frac{1}{2} \frac{d}{dt} \left(\|u\|_{\dot{H}^{\alpha/2}}^2 + \|q\|_{\dot{H}^{\alpha/2}}^2 \right) + \|u\|_{\dot{H}^\alpha}^2 \leq - \int_{\mathbb{T}} [\Lambda^\alpha, u] q \partial_x u dx.$$

We define

$$K = - \int_{\mathbb{T}} [\Lambda^\alpha, u] q \partial_x u dx.$$

Using the classical Kenig-Ponce-Vega commutator estimate [10] and Sobolev embedding, we have that

$$\begin{aligned} \|[\Lambda^\alpha, u] q\|_{L^2} &\leq c \left(\|\partial_x u\|_{L^{2+\epsilon}} \|\Lambda^{\alpha-1} q\|_{L^{\frac{4+2\epsilon}{\epsilon}}} + \|\Lambda^\alpha u\|_{L^2} \|q\|_{L^\infty} \right) \\ (8) \quad &\leq c \left(\|u\|_{\dot{H}^{1+\frac{\epsilon}{4+2\epsilon}}} \|\Lambda^{\alpha-1} q\|_{\dot{H}^{\frac{1}{2}-\frac{\epsilon}{4+2\epsilon}}} + \|u\|_{\dot{H}^\alpha} \|q\|_{\dot{H}^{\frac{\alpha}{2}}} \right). \end{aligned}$$

Thus, taking ϵ such that

$$1 + \frac{\epsilon}{4+2\epsilon} = \alpha, \quad i.e. \quad \epsilon = \frac{4\alpha-4}{3-2\alpha}$$

Equation (8) reads

$$(9) \quad \|[\Lambda^\alpha, u] q\|_{L^2} \leq c \left(\|u\|_{\dot{H}^\alpha} \|q\|_{\dot{H}^{\frac{1}{2}}} + \|u\|_{\dot{H}^\alpha} \|q\|_{\dot{H}^{\frac{\alpha}{2}}} \right)$$

Using (9) and Poincaré inequality, we have that

$$K \leq c \|u\|_{\dot{H}^\alpha}^2 \|q\|_{\dot{H}^{\frac{\alpha}{2}}}$$

Then, we have that

$$\frac{d}{dt} E_{\frac{\alpha}{2}} + D_{\frac{\alpha}{2}} \leq c \sqrt{E_{\frac{\alpha}{2}}} D_{\frac{\alpha}{2}}.$$

Thus, due to the smallness restriction on the initial data, we obtain

$$E_{\frac{\alpha}{2}}(t) + \delta \int_0^t D_{\frac{\alpha}{2}}(s) ds \leq E_{\frac{\alpha}{2}}(0)$$

for $0 < \delta$ small enough.

Step 2; H^1 estimate: Our starting point is (6). Then we use the interpolation

$$\|g\|_{\dot{H}^{0.25}}^2 \leq c \|g\|_{L^2} \|g\|_{\dot{H}^{0.5}},$$

to obtain

$$I \leq c \|q\|_{\dot{H}^1} \|u\|_{\dot{H}^1} \|u\|_{\dot{H}^{1.5}} \leq c E_1 D_{\frac{\alpha}{2}} + \frac{D_1}{2}.$$

Collecting all the estimates, we have that

$$\frac{d}{dt} E_1 + D_1 \leq c E_1 D_{\frac{\alpha}{2}},$$

and we conclude using Gronwall's inequality. The H^2 estimates follows as in the proof of Theorem 1.

5. PROOF OF COROLLARY 1

Using $\alpha \geq 0.5$ and the estimate (6), we have that

$$I \leq \frac{3}{2}C_S \|q\|_{\dot{H}^1} \|u\|_{\dot{H}^{1+\alpha/2}}^2 \leq \frac{3}{2}C_S \sqrt{E_1} D_1.$$

Thus,

$$\frac{1}{2} \frac{d}{dt} E_1 + D_1 \leq \frac{3}{2}C_S \sqrt{E_1} D_1.$$

Thus, due to the smallness restriction on the initial data, we obtain

$$E_1(t) + \delta \int_0^t D_1(s) ds \leq E_1(0)$$

for $0 < \delta$ small enough. Equipped with this estimates, we can repeat the argument as in Step 2 in Theorem 1.

6. PROOF OF THEOREM 3

Recall that the condition

$$\operatorname{curl} q_0 = 0$$

propagates in time, *i.e.*

$$\operatorname{curl} q(t) = \operatorname{curl} q_0 + \frac{1}{2} \int_0^t \operatorname{curl} \nabla u^2 ds = 0.$$

Using Plancherel Theorem, we have that

$$\begin{aligned} \|\nabla q\|_{L^2}^2 &= C \sum_{\xi \in \mathbb{Z}^2} |\xi|^2 |\hat{q}(\xi)|^2 \\ &= C \sum_{\xi \in \mathbb{Z}^2} (\xi_1^2 + \xi_2^2) (\hat{q}_1^2 + \hat{q}_2^2). \end{aligned}$$

Due to the irrotationality

$$\xi^\perp \cdot \hat{q} = 0.$$

Then, we compute

$$\begin{aligned} \|\nabla \cdot q\|_{L^2}^2 &= C \sum_{\xi \in \mathbb{Z}^2} |\xi \cdot \hat{q}(\xi)|^2 \\ &= C \sum_{\xi \in \mathbb{Z}^2} (\xi_1 \hat{q}_1(\xi) + \xi_2 \hat{q}_2(\xi))^2 \\ &= C \sum_{\xi \in \mathbb{Z}^2} (\xi_1 \hat{q}_1(\xi))^2 + (\xi_2 \hat{q}_2(\xi))^2 + 2\xi_1 \hat{q}_1(\xi) \xi_2 \hat{q}_2(\xi) \\ &= C \sum_{\xi \in \mathbb{Z}^2} (\xi_1 \hat{q}_1(\xi))^2 + (\xi_2 \hat{q}_2(\xi))^2 + (\xi_2 \hat{q}_1(\xi))^2 + (\xi_1 \hat{q}_2(\xi))^2. \end{aligned}$$

So, the vector field q satisfies

$$\|\nabla q\|_{L^2} \leq \|\nabla \cdot q\|_{L^2}.$$

As a consequence of $\langle \partial_t q_i \rangle = 0$ and $\langle q_0 \rangle = 0$, every coordinate of q satisfy

$$\langle q_i(t) \rangle = 0,$$

and the Poincaré-type inequality

$$(10) \quad \|q\|_{L^2} \leq c \|\nabla \cdot q\|_{L^2}.$$

Notice that in two dimensions we also have the energy balance (5). We test equation (1) against $\Lambda^2 u$ and use the equation for q . We obtain

$$\frac{1}{2} \frac{d}{dt} \left(\|u\|_{\dot{H}^1}^2 + \|\nabla \cdot q\|_{L^2}^2 \right) = -\|u\|_{\dot{H}^2}^2 - \int_{\mathbb{T}^2} \nabla u \cdot q \Delta u dx + \int_{\mathbb{T}^2} |\nabla u|^2 \nabla \cdot q dx.$$

Using Hölder inequality, Sobolev embedding and interpolation, we have that

$$\begin{aligned} \frac{d}{dt} \left(\|u\|_{\dot{H}^1}^2 + \|\nabla \cdot q\|_{L^2}^2 \right) + 2\|u\|_{\dot{H}^2}^2 &\leq c \left(\|u\|_{\dot{H}^{1.5}} \|q\|_{L^4} \|u\|_{\dot{H}^2} + \|u\|_{\dot{H}^{1.5}}^2 \|\nabla \cdot q\|_{L^2} \right) \\ &\leq c \|u\|_{\dot{H}^1}^{0.5} \|q\|_{L^2}^{0.5} \|q\|_{\dot{H}^1}^{0.5} \|u\|_{\dot{H}^2}^{1.5} \\ &\quad + c \|u\|_{\dot{H}^1} \|u\|_{\dot{H}^2} \|\nabla \cdot q\|_{L^2}. \end{aligned}$$

Using the Höodge decomposition estimate together with the irrotationality of q and (10), we have that

$$(11) \quad \|q\|_{H^1} \leq c (\|q\|_{L^2} + \|\nabla \cdot q\|_{L^2}) \leq c \|\nabla \cdot q\|_{L^2}.$$

Due to (5), we obtain that

$$\|q\|_{L^\infty(0,\infty,L^2)}^2 + \|u\|_{L^2(0,\infty,\dot{H}^1)}^2 \leq C(\|u_0\|_{L^2}, \|q_0\|_{L^2})$$

so,

$$\frac{d}{dt} \left(\|u\|_{\dot{H}^1}^2 + \|\nabla \cdot q\|_{L^2}^2 \right) + \|u\|_{\dot{H}^2}^2 \leq c \|u\|_{\dot{H}^1}^2 \|\nabla \cdot q\|_{L^2}^2.$$

Using Gronwall's inequality and the integrability of $\|u\|_{\dot{H}^1}^2$ (see (5)), we obtain

$$\begin{aligned} E_1 &\leq C(\|u_0\|_{H^1}, \|q_0\|_{H^1}), \\ \int_0^T D_1(s) ds &\leq C(\|u_0\|_{H^1}, \|q_0\|_{H^1}, T), \quad \forall 0 < T < \infty. \end{aligned}$$

To obtain the H^2 estimates, we test against $\Lambda^4 u$. Then, using the previous H^1 uniform bound and

$$\|q\|_{L^\infty}^2 \leq c \|q\|_{L^2} \|q\|_{H^2} \leq c \|q\|_{L^2} \|\Delta q\|_{L^2},$$

we have that

$$\begin{aligned} (12) \quad \frac{1}{2} \frac{d}{dt} \|\Delta u\|_{L^2}^2 + \|u\|_{\dot{H}^3}^2 &= - \int_{\mathbb{T}^d} \nabla \Delta u \nabla (\nabla u \cdot q) dx \\ &\quad - \int_{\mathbb{T}^d} u \nabla \Delta u \cdot \nabla (\nabla \cdot q) dx - \int_{\mathbb{T}^d} \nabla u \cdot \nabla \Delta u \nabla \cdot q dx. \end{aligned}$$

Due to the irrotationality of q and the identity

$$\nabla \nabla \cdot q - \Delta q = \text{curl}(\text{curl } q),$$

we have

$$\partial_t \Delta q = \partial_t \nabla (\nabla \cdot q) = \nabla |\nabla u|^2 + u \nabla \Delta u + \nabla u \Delta u.$$

Applying Sobolev embedding and interpolation, we obtain that (12) can be estimated as

$$\frac{d}{dt} \left(\|\Delta u\|_{L^2}^2 + \|\Delta q\|_{L^2}^2 \right) + \|u\|_{\dot{H}^3}^2 \leq c \|\Delta q\|_{L^2}^2,$$

so,

$$\frac{d}{dt}E_2 + D_2 \leq cE_2,$$

and we conclude using Gronwall's inequality.

7. PROOF OF COROLLARY 2

We test the equation (1) against $\Lambda^2 u$. We obtain that

$$\frac{1}{2} \frac{d}{dt} \left(\|u\|_{\dot{H}^1}^2 + \|\nabla \cdot q\|_{L^2}^2 \right) = -\|u\|_{\dot{H}^{1+\frac{\alpha}{2}}}^2 + \int_{\mathbb{T}^2} \nabla(\nabla u \cdot q) \nabla u dx + \int_{\mathbb{T}^2} |\nabla u|^2 \nabla \cdot q dx.$$

After a short computation, using Hölder estimates, Sobolev embedding and interpolation, we obtain that

$$\frac{1}{2} \frac{d}{dt} \left(\|u\|_{\dot{H}^1}^2 + \|\nabla \cdot q\|_{L^2}^2 \right) + \|u\|_{\dot{H}^{1+\frac{\alpha}{2}}}^2 \leq c \|u\|_{\dot{H}^{1.5}}^2 \|q\|_{\dot{H}^1}.$$

Using (11) and $\alpha \geq 0$, we obtain

$$\frac{1}{2} \frac{d}{dt} \left(\|u\|_{\dot{H}^1}^2 + \|\nabla \cdot q\|_{L^2}^2 \right) + D_1 \leq c D_1 \|\nabla \cdot q\|_{L^2}^2.$$

We conclude the result with the previous ideas.

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E-mail address: granero@math.univ-lyon1.fr

UNIV LYON, UNIVERSITÉ CLAUDE BERNARD LYON 1, CNRS UMR 5208, INSTITUT CAMILLE JORDAN, 43 BLVD. DU 11 NOVEMBRE 1918, F-69622 VILLEURBANNE CEDEX, FRANCE.